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MINIMAL NON-QUATERNION-FREE FINITE 2-GROUPS

BY

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ABSTRACT

We classify minimal non-quaternion-free finite 2-groups which possess a nonmodular proper subgroup.

Here we classify minimal non-quaternion-free finite 2-groups. A 2-group G is minimal non-quaternion-free if G is not quaternion-free but each proper subgroup of G is quaternion-free. We recall that a 2-group G is modular if and only if G is D_8-free. We consider here only finite 2-groups and our notation is standard.

THEOREM 1.1: *Let G be a minimal non-quaternion-free 2-group. Then G possesses a unique normal subgroup N such that $G/N \cong Q_8$. We have $N < \Phi(G)$ and so $G/\Phi(G) \cong E_4$ and $\Omega_1(G) \le \Phi(G)$. If R is any G-invariant subgroup of index 2 in N, then $X = G/R$ is the minimal nonabelian metacyclic group of order 2^4 and exponent 4:*

$$X = \langle a, b | a^4 = b^4 = 1, a^b = a^{-1} \rangle,$$

where $X/\langle a^2b^2 \rangle \cong Q_8$, $X/\langle b^2 \rangle \cong D_8$, and $X/\langle a^2 \rangle \cong C_4 \times C_2$. In particular, if $|G| > 8$, then G has a normal subgroup S such that $G/S \cong D_8$ and so G is nonmodular.

Proof: The group G possesses a normal subgroup N such that $G/N \cong Q_8$. Clearly, $N < \Phi(G)$ and so $d(G) = 2$. For each $x \in G - \Phi(G)$, $x^2 \in \Phi(G) - N$ and so $o(x) \ge 4$. Consequently, $\Omega_1(G) \le \Phi(G)$. We may assume $|G| > 8$ and let R be any G-invariant subgroup of index 2 in N.

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We shall determine the structure of G/R. For that purpose we may assume $R = \{1\}$ so that $|N| = 2$, $N < \Phi(G)$, $N \leq Z(G)$, $|\Phi(G)| = 4$, and $\Phi(G)/N = Z(G/N) = (G/N)'$, where $G/N \cong Q_8$. Let $M_1/N \neq M_2/N$ be two distinct cyclic subgroups of order 4 in G/N. Then M_1 and M_2 are abelian and $M_1 \cap M_2 = \Phi(G)$. We get $C_G(\Phi(G)) \geq \langle M_1, M_2 \rangle = G$ and so $\Phi(G) = Z(G)$. Each maximal subgroup of G is abelian and therefore G is minimal nonabelian, which implies $|G'| = 2$. But G' covers $\Phi(G)/N$ and so $\Phi(G) = N \times G' \cong E_4$. For each $x \in G - \Phi(G)$, $x^2 \in \Phi(G) - N$ and so $\exp(G) = 4$ and $\Omega_1(G) = \Phi(G)$. Since $\Phi(G) = \mho_1(G)$ and the involution in N is not a square in G, each involution in $\Phi(G) - N$ must be a square in G. Hence, there is $a \in G - \Phi(G)$ with $\langle a^2 \rangle = G'$ (which implies that $\langle a \rangle$ is normal in G) and $b \in G - \Phi(G)$ with $b^2 \notin G'$. Hence $\langle a \rangle \cap \langle b \rangle = \{1\}$ and so $|\langle a, b \rangle| = |\langle a \rangle||\langle b \rangle| = 2^4$, which implies $G = \langle a, b \rangle$. In particular, $[a, b] \neq 1$ and $a^b = a^{-1}$ and we are done.

It remains to show that the normal subgroup N with $G/N \cong Q_8$ is unique. Indeed, let $M \neq N$ be another normal subgroup of G with $G/M \cong Q_8$. Since $|\Phi(G) : N| = |\Phi(G) : M| = 2$, M and N are two distinct maximal subgroups of $\Phi(G)$ and therefore $D = M \cap N$ is normal in G and $|N : D| = 2$. By the above, $X = G/D$ is the (unique) minimal nonabelian metacyclic group of order 2^4 and exponent 4, where G/M and G/N are two distinct epimorphic images of G/D of order 8 and both are isomorphic to Q_8. On the other hand, we know that X has exactly three factor groups of order 8 and they are isomorphic to Q_8, D_8, and $C_4 \times C_2$. This contradiction shows that N is unique. ∎

Let G be a minimal non-Q_8-free 2-group of order > 8 such that each proper subgroup of G is modular. Then G is a minimal nonmodular 2-group and such groups have been classified in [1]. Therefore, we may assume in the sequel that G has a maximal subgroup H which is nonmodular. Since H is Q_8-free, we are in a position to apply Theorem 1.7 in [2] classifying nonmodular Q_8-free 2-groups. Also, we assume that the reader is familiar with Propositions 1.8, 1.9, and 1.10 in [2] describing the Wilkens groups of types (a), (b), and (c), which appear in Theorem 1.7 in [2].

THEOREM 1.2: *Let G be a minimal non-quaternion-free 2-group which has a nonmodular proper subgroup. Then G has one of the following properties:*

(i) $\Omega_1(G) = \Phi(G) = A\langle t \rangle$, *where A is a maximal normal abelian subgroup of G with $\exp(A) > 2$, t is an involution inverting each element in A, and $G/A \cong Q_8, D_8$, or $C_4 \times C_2$. If $G/A \cong D_8$ or $C_4 \times C_2$, then A is abelian of type $(4, 2, \ldots, 2)$.*

(ii) $\Omega_1(G) = EE_1$, *where $E \neq E_1$ are the only maximal normal elementary* *abelian subgroups of* $\Omega_1(G)$ *and* $|\Omega_1(G) : E| = 2$, $|\Omega_1(G) : E_1| \geq 2$. *We have either* $\Omega_1(G) = \Phi(G)$ *(and then* $G/\Omega_1(G) \cong E_4$*) or* $G/\Omega_1(G) \cong Q_8$. *If* $G/\Omega_1(G) \cong Q_8$, *then also* $|\Omega_1(G) : E_1| = 2$, *which implies* $\Omega_1(G) \cong D_8 \times E_{2^s}$.

(iii) $\Omega_1(G) = E$ *is elementary abelian and* G/E *is isomorphic to* Q_8, M_{2^n}, $n \geq 4$, *or* $C_{2^m} \times C_2, m \geq 1$. *If* $G/E \cong M_{2^n}$ *or* $C_{2^m} \times C_2$ *with* $m \geq 2$, *then* $\Omega_2(G)$ *is abelian of type* $(4, 4, 2, \ldots, 2)$.

Proof: Let G be a minimal non-quaternion-free 2-group possessing a maximal subgroup H which is nonmodular. It follows that H is a Wilkens group of type (a), (b), or (c). In particular, $H/\Omega_1(H)$ is cyclic, where $\Omega_1(H) = \Omega_1(G)$ since $\Omega_1(G) \leq \Phi(G)$. If $\Omega_1(G)$ is not elementary abelian, then we know (by the structure of the Wilkens group H) that $\Omega_1(G)$ is nonmodular and so in that case each maximal subgroup M of G is a Wilkens group. It follows that $M/\Omega_1(G)$ is cyclic, where $\Omega_1(M) = \Omega_1(G)$. In that case, $X = G/\Omega_1(G) \cong E_4$ or Q_8. Here we have used a trivial fact that a noncyclic 2-group X all of whose maximal subgroups are cyclic must be isomorphic to E_4 or Q_8.

(i) Suppose that H is a Wilkens group of type (a). Then H is a semidirect product $H = \langle x \rangle \cdot A$, where A is a maximal normal abelian subgroup of H with $\exp(A) > 2$ and, if t is the involution in $\langle x \rangle$, then t inverts each element of A. We have $\Omega_1(H) = \Omega_1(G) = A\langle t \rangle$ and A is a characteristic subgroup in H (Proposition 1.8 in [2]) and so A is normal in G. By the previous paragraph, $G/\Omega_1(G) \cong E_4$ or Q_8. However, if $G/\Omega_1(G) \cong Q_8$, then G/A (having a cyclic subgroup H/A of index 2) is of maximal class. But such a group does not have a proper factor group isomorphic to Q_8. Hence $G/\Omega_1(G) \cong E_4$ and so $\Omega_1(G) = \Phi(G)$. Since $d(G/A) = 2$, we get $G/A \cong Q_8, D_8$ or $C_4 \times C_2$.

Suppose that A is not a maximal normal abelian subgroup of G. Let B be a maximal normal abelian subgroup of G containing A so that $B \cap H = A$, $|B : A| = 2$, and $G = \langle x \rangle \cdot B$ with $\langle x \rangle \cap B = \{1\}$. Let $b \in B - A$ and we compute

$$(bb^t)^t = b^t b = bb^t \in A,$$

since $b^t \in B - A$ so that $bb^t = s \in \Omega_1(A)$. If $s = 1$, $b^t = b^{-1}$ and so t inverts each element of the abelian group B. It follows that $G = \langle x \rangle \cdot B$ is a Wilkens group of type (a). In that case G would be Q_8-free (Proposition 1.8 in [2]), a contradiction. Hence $s \neq 1$ and

$$(bt)^2 = btbt = bb^t = s$$

shows that $o(bt) = 4$. Let a be an element of order 4 in A. We have $a^{bt} = a^{-1}$ and $\langle a, bt \rangle \neq G$ (since $a \in \Phi(G)$) and so $\langle a, bt \rangle$ is Q_8-free, contrary to Lemma 1.1 in [2]. We have proved that A is a maximal normal abelian subgroup of G.

Suppose that $G/A \cong D_8$ or $C_4 \times C_2$. In that case there is a maximal subgroup K of G such that $K/A \cong E_4$ and $\Omega_1(K) = \Omega_1(G)$. Then K is a Wilkens group of type (a) or (b) since $|K : \Omega_1(K)| = 2$ (and so K cannot be a Wilkens group of type (c)). Suppose that K is a Wilkens group of type (a) with respect to a maximal normal abelian subgroup A_1 of K with $\exp(A_1) > 2$. We know that $A_1 \leq \Omega_1(K) = \Omega_1(G)$, $|\Omega_1(G) : A_1| = 2$ and K/A_1 is cyclic. Since K/A is noncyclic, we have $A_1 \neq A$. All elements in $A_1 - A$ are involutions and, if $t_0 \in A_1 - A$, then t_0 inverts and centralizes each element in $A \cap A_1$ and so $A \cap A_1$ is elementary abelian. It follows that $\exp(A_1) = 2$, a contradiction. We have proved that K is a Wilkens group of type (b) with respect to a maximal normal elementary abelian subgroup E of K. We know that $|\Omega_1(K) : E| = 2$ (Proposition 1.9 in [2]). Since $\Omega_1(K) = \Omega_1(G)$, we have $|A : A \cap E| = 2$ and so A is abelian of type $(4, 2, \dots, 2)$.

(ii) Assume that H is a Wilkens group of type (b) with respect to E and $|\Omega_1(H) : E| = 2$, where H/E is cyclic. We have $\Omega_1(H) = \Omega_1(G)$ and $\Omega_1(G) \leq \Phi(G)$ so that $\Omega_1(G)$ is nonmodular. Indeed, $\Omega_1(G)$ has exactly two maximal normal elementary abelian subgroups E and E_1, where $\Omega_1(G) = EE_1$, and so $\Omega_1(G)$ is a Wilkens group of type (b) (and so nonmodular). By the first paragraph of the proof, $G/\Omega_1(G) \cong E_4$ or Q_8.

Suppose that $G/\Omega_1(G) \cong Q_8$. It is easy to see that E is not normal in G. Suppose false. Since G/E has a cyclic subgroup H/E of index 2 and $G/\Omega_1(G) \cong Q_8$, we get that G/E must be of maximal class. But there is no 2-group of maximal class having Q_8 as a proper homomorphic image. Hence E is not normal in G and so for each $x \in G - H$, $E^x = E_1$. In particular, $|\Omega_1(G) : E_1| = 2$ and $F = E \cap E_1 = Z(\Omega_1(G))$. Take $e \in E - F$, $e_1 \in E_1 - F$ so that $D = \langle e, e_1 \rangle \cong D_8$ and, if V is a complement of $\langle [e, e_1] \rangle$ in F, then $\Omega_1(G) = D \times V \cong D_8 \times E_{2^s}$.

(iii) Suppose that H is a Wilkens group of type (b) with respect to E and $E = \Omega_1(H)$ so that $E = \Omega_1(G)$ (since $E \leq \Phi(G)$), E is normal in G and G/E is noncyclic with the cyclic subgroup H/E of index 2.

Let F/E be a proper subgroup of G/E such that $F/E \cong E_4$. Then F is abelian. Indeed, since $F \neq G$, F is Q_8-free. If F is not D_8-free, then F must be a Wilkens group. But then $F/\Omega_1(F)$ must be cyclic. This is a contradiction since $\Omega_1(F) = E$. It follows that F is D_8-free. Since $\exp(F) \leq 4$, F must be abelian (see [3]).

Since each proper subgroup of G/E is Q_8-free, G/E cannot be semidihedral or Q_{2^m} with $m \geq 4$. Suppose that $G/E \cong D_{2^n}$, $n \geq 3$. In that case G/E is generated by its four-subgroups and so $E \leq Z(G)$. This is a contradiction, because in that case H would be abelian (noting that H/E is cyclic). We have proved that G/E is isomorphic to Q_8, M_{2^n}, $n \geq 4$, or $C_{2^m} \times C_2$, $m \geq 1$.

Suppose that G/E is not isomorphic to Q_8 or $C_2 \times C_2$. Set $F/E = \Omega_1(G/E)$ so that $F/E \cong E_4$. By the above, F is abelian. Obviously, $F = \Omega_2(G)$ is abelian of type $(4,4,2,\dots,2)$.

(iv) Finally, assume that H is a Wilkens group of type (c). We have $\Omega_1(H) = \Omega_1(G) \cong D_8 \times E_{2^s}$ and $H/\Omega_1(G)$ is cyclic of order ≥ 4 (Proposition 1.10 in [2]). The subgroup $\Omega_1(G)$ has exactly three abelian maximal subgroups E_1, E_2, A, where E_1 and E_2 are elementary abelian and A is abelian of type $(4,2,\dots,2)$. By the structure of H, E_1 and E_2 are not normal in H and $H/E_1 \cap E_2 \cong M_{2^n}$, $n \geq 4$. By the first paragraph of the proof, $G/\Omega_1(G) \cong Q_8$ since $|H/\Omega_1(G)| \geq 4$. Thus $H/E_1 \cap E_2 \cong M_{2^4}$.

On the other hand, $\Omega_1(G)$ is normal in G and so $N_G(E_1) = N_G(E_2) = K$ is a maximal subgroup of G distinct from H. Since $K \geq \Omega_1(G)$, K is nonmodular and therefore K is a Wilkens group. But K cannot be a Wilkens group of type (c) since E_1 and E_2 are normal in K. Hence K is either a Wilkens group of type (a) with respect to A or K is a Wilkens group of type (b) with respect to E_1 or E_2. The group G with such a maximal subgroup K has been considered in (i) and (ii) and so we do not get here new possibilities for the structure of G. Our theorem is proved. ∎

References

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